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MAXIMUM PRINCIPLES FOR MARTINGALE RANDOM FIELDS VIA NON-ANTICIPATING STOCHASTIC DERIVATIVES

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ABSTRACT. We find a maximum principle for processes driven by martingale random fields. We do so by describing the adjoint processes with non-anticipating stochastic derivatives. In the case of the Levy processes this mimics maximum principles with Malliavin derivatives, but we replace Malliavin differentiability conditions with L_2-conditions.

As an application we use the maximum principle to solve a portfolio optimization problem for assets with credit risk modeled by doubly stochastic Poisson processes.

1. INTRODUCTION

We find a maximum principle for an optimization problem when the state process depends on a martingale random field [CW75, DE10], a generalization of the martingale. We do so in a pertubation-based approach, using the *non-anticipating stochastic derivative* [Di 02, Di 03, DE10] to describe the adjoint processes.

We consider the performance functional

$$J(u) = \mathbb{E}\Big[\int_0^T f_t(u_t, X_t)\, dt + g(X_T)\Big] \tag{1.1}$$

and the associated optimal stochastic control problem, where u is the control and the state process is given by the semi-martingale X,

$$X_t^{(u)} = X_0 + \int_0^t b_s(u_s, X_s)\, ds + \int_0^t \int_{\mathcal{Z}} \phi_s(z, u_s, X_s)\, \mu(ds, dz), \quad t \in [0, T],$$

where the last integral is over the martingale random field μ on $[0, T] \times \mathcal{Z}$.

The goal is to find $\sup_u J(u)$ for controls adapted to the filtration $\mathbb{F}$, where X is adapted to the filtration $\mathbb{G}$ and $\mathbb{F} \subseteq \mathbb{G}$. This is a problem with

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partial information if X is not $\mathbb{F}$-adapted. We find (candidates for) optimal solutions by investigating

$$\frac{\partial}{\partial y} J(u + y\beta)\big|_{y=0}, \quad u, \beta \text{ admissible controls and } y \in \mathbb{R}.$$

Key to our approach is the non-anticipating derivative $\mathscr{D}$, an operator from the probability space $L_2(\Omega, \mathcal{G}, \mathbb{P})$ to the space of integrable random fields. The operator $\mathscr{D}$ coincides with the dual of the Itô non-anticipating stochastic integral with respect to a general martingale random field. Indeed we have that, for $\xi \in L_2(\Omega, \mathcal{G}, \mathbb{P})$, that

$$\mathbb{E}\Big[\xi \int_0^T \int_{\mathcal{Z}} \kappa(s,z)\, \mu(ds,dz)\Big] = \mathbb{E}\Big[\int_0^T \int_{\mathcal{Z}} (\mathscr{D}_{s,z}\xi)\kappa(s,z)\, \Lambda(ds,dz)\Big].$$

Here Λ represents the conditional variance measure associated to μ. These concepts are further detailed in the forthcoming sections 2 and 3.

Maximum principles using the duality relation of the Malliavin derivative with the Skorohod integral have been studied in [DNØ09, MBØZ12]. This limits the method to Lévy processes and restrictions of the method are imposed to match the domains of the Malliavin derivative. The key novelty in our paper is the use of the non-anticipating stochastic derivative, which enables us to treat very general martingale noises. Furthermore, in the case of Lévy noise, we reduce assumptions of Malliavin differentiable random variables to square integrability. Since the non-anticipating derivative coincide with the Malliavin derivative when both are well defined, this extends previous results. An additional benefit of the martingale random field structure is an easy extension to multi-dimensional controls.

For the portfolio problem with default risk, the main result is extended to a simpler sufficient condition for optimal control.

The benefit of the duality type approach used herein compared to HJB-type equations are that we can treat problems of partial information, which are of non-Markovian nature. Maximum principles for partial information has also been studied in a BSDE approach in e.g. [BØ07, AØ12] but again limited to the case of Lévy noise. Note that partial information in the sense of the filtrations $\mathbb{F}$ and $\mathbb{G}$ differs from *partial observation* problems with noisy observations of the state process as treated in e.g [Ben92, KX91, Tan98].

In this paper, the maximum principle is studied in section 3. The specific discussion of a Lévy type martingale μ is considered in section 4. Section 5 presents an application to portfolio optimization in a market with assets subject to default risk.

2. The martingale random field

We will define integration and the non-anticipating stochastic derivative over a martingale random field μ. We refer to [DE10] for a detailed discussion on these concepts.

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space equipped with a right-continuous filtration $\mathbb{G} := \{\mathcal{G}_t,\, t \in [0,T]\}$. Let $\mathcal{Z}$ be a separable topological space. We denote $\mathcal{B}_{\mathcal{Z}}$ as the Borel σ-algebra on $\mathcal{Z}$ and $\mathcal{B}_{[0,T]\times\mathcal{Z}}$ as the Borel σ-algebra on the product space $[0,T]\times\mathcal{Z}$. Note that $\mathcal{B}_{[0,T]\times\mathcal{Z}}$ is generated by a semi-ring of sets of type

$$\Delta = (t,s] \times Z, \quad 0 \le t < s \le T,\, Z \in \mathcal{B}_{\mathcal{Z}}.$$

We say that the stochastic set function $\mu(\Delta)$, $\Delta \in \mathcal{B}_{[0,T]\times\mathcal{Z}}$ is a martingale random field on $[0,T]\times\mathcal{Z}$ (with square integrable values) with respect to $\mathbb{G}$ if it satisfies the following properties [DE10, Definition 2.1]:

i) μ has a tight, σ-finite variance measure $V(\Delta) = E\big[\mu(\Delta)^2\big]$, $\Delta \in \mathcal{B}_{[0,T]\times\mathcal{Z}}$, which satisfies $V(\{0\}\times\mathcal{Z}) = 0$.

ii) μ is additive, i.e. for pairwise disjoint sets $\Delta_1, \dots, \Delta_K$: $V(\Delta_k) < \infty$

$$\mu\Big(\bigcup_{k=1}^{K} \Delta_k\Big) = \sum_{k=1}^{K} \mu(\Delta_k)$$

and σ-additive in L_2.

iii) μ is $\mathbb{G}$-adapted.

iv) μ has the *martingale property.* Consider $\Delta \subseteq (t,T]\times\mathcal{Z}$. we have:

$$\mathbb{E}\Big[\mu(\Delta)\,\Big|\,\mathcal{G}_t\Big] = 0.$$

v) μ has conditionally orthogonal values. For any $\Delta_1, \Delta_2 \subseteq (t,T]\times\mathcal{Z}$ such that $\Delta_1 \cap \Delta_2 = \emptyset$ we have:

$$\mathbb{E}\Big[\mu(\Delta_1)\mu(\Delta_2)\,\Big|\,\mathcal{G}_t\Big] = 0.$$

In particular, any finite sums of orthogonal, square integrable martingales would be a martingale random field in the sense of i)-ii)-iii)-iv)-v) above. In general, the filtration $\mathbb{G}$ does not have to be the one generated by μ.

The $\mathbb{G}$-predictable σ-algebra on $\Omega\times[0,T]\times\mathcal{Z}$ is denoted by $\mathcal{P}_{[0,T]\times\mathcal{Z}}$ and is generated by sets of type

$$\Delta = A \times (t,s] \times Z, \quad 0 \le t < s \le T,\, Z \in \mathcal{B}_{\mathcal{Z}},\, A \in \mathcal{G}_s.$$

The $\mathbb{G}$-predictable σ-algebra $\Omega\times[0,T]$ is denoted by $\mathcal{P}_{[0,T]}$ and is generated by sets of type $\Delta = A \times (t,s]$, $0 \le t < s \le T, A \in \mathcal{G}_s$. On $(\Omega\times[0,T]\times\mathcal{Z}, \mathcal{P}_{[0,T]\times\mathcal{Z}})$ the random field μ has a σ-finite conditional random variance measure [DE10, Theorem 2.1]. For martingale processes the conditional variance measure is the $\mathbb{G}$-predictable compensator. We denote this conditional variance measure by Λ, and it has the following properties

$$\mathbb{E}\big[\,\mu(\Delta)^2\big|\mathcal{G}_t\big] = \Lambda(\Delta), \quad \text{in } L_1(\Omega,\mathcal{F},\mathbb{P}) \text{ for } \Delta \subseteq (t,T]\times\mathcal{Z},$$
$$\mathbb{E}\big[\mu(\Delta)^2\big] = \mathbb{E}\big[\Lambda(\Delta)\big].$$

For later purposes we assume that Λ is absolutely continuous with respect to the Lebesgue measure on $[0,T]$. Namely we assume that there exists a transition kernel λ from $(\Omega \times [0,T], \mathcal{P}_{[0,T]})$ to $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ such that $\Lambda(\omega, dt, dz) = \lambda_t(\omega, dz)\, dt$. Meaning that the mapping $(\omega, t) \to \lambda_t(\omega, Z)$ is $\mathcal{P}_{[0,T]}$ measurable for every $Z \in \mathcal{B}_{\mathcal{Z}}$ and $\lambda_t(\omega, \cdot)$ is measure on $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ for every $(\omega, t) \in \Omega \times [0,T]$. We refer to [Çın11] for further details on transition kernels.

We denote $\mathcal{I}$ as the set of $\mathbb{G}$-predictable random fields $\phi : \Omega \times [0,T] \times \mathcal{Z} \to \mathbb{R}$ satisfying

$$\|\phi\|_{\mathcal{I}} := \mathbb{E}\Big[\int_0^T \int_{\mathcal{Z}} \phi^2(s,z)\, \lambda_s(dz) ds\Big]^{\frac{1}{2}} < \infty.$$

We say that $\phi \in \mathcal{I}$ is a simple random field if it can be written as

$$\phi(s,z,\omega) = \sum_{i=1}^{N} \phi_i(\omega) \mathbf{1}_{\Delta_i}(s,z), \quad \Delta_i \in \mathcal{B}_{[0,T]\times\mathcal{Z}}, \tag{2.1}$$

where ϕ_i are bounded random variables (for $i = 1, \dots N < \infty$). Simple, $\mathbb{G}$-predictable random fields are dense in $\mathcal{I}$ by the usual Itô integration type arguments and we have, for every $\phi \in \mathcal{I}$:

$$\begin{aligned} \mathbb{E}\Big[\big(\int_0^T \int_{\mathcal{Z}} \phi(s,z)\, \mu(ds,dz)\big)^2\Big] &= \mathbb{E}\Big[\int_0^T \int_{\mathcal{Z}} \phi^2(s,z)\, \Lambda(ds,dz)\Big] \\ &= \mathbb{E}\Big[\int_0^T \int_{\mathcal{Z}} \phi^2(s,z)\, \lambda_s(dz) ds\Big]. \end{aligned} \tag{2.2}$$

Remark also that $\phi \in \mathcal{I}$ implies that

$$\int_{\mathcal{Z}} \phi_t(z)^2\, \lambda_t(dz) < \infty, \quad dt \times d\mathbb{P}\text{a.e.}$$

3. Duality relations with non-anticipating stochastic derivatives

The *non-anticipating stochastic derivative* is a characterization of the integrand in the Kunita-Watanabe decomposition, developed in [Di 02, Di 03, DR07, Di 07, DE10]. It is the adjoint (linear) operator $\mathscr{D} = I^*$ of the stochastic integral:

$$\mathscr{D} : L_2(\Omega, \mathcal{G}, \mathbb{P}) \Longrightarrow \mathcal{I}.$$

A full characterization is given in constructive form using the elements of the following dissecting system, a sequence of partitions of $[0,T] \times \mathcal{Z}$. Let $A_n \subseteq [0,T] \times \mathcal{Z}$ be an increasing sequence of Borel-sets such that $V(A_n) < \infty$

for all $n \in \mathbb{N}$ and $\cup_{n=1}^{\infty} A_n = [0,T] \times \mathcal{Z}$. For every n we chose a partition $\{\Delta_{n,k}\}$, $k = 1, \dots, K_n < \infty$, of A_n such that[1]

$$\bigcup_{1 \le k \le K_n} \Delta_{n,k} = A_n, \tag{3.1}$$

$$\Delta_{n,k} = (s_{n,k}, t_{n,k}] \times Z_{n,k}, \quad 0 \le s_{n,k} < t_{n,k} \le T,\ Z_{n,k} \in \mathcal{B}_{\mathcal{Z}} \tag{3.2}$$

$$\max_{1 \le k \le K_n} t_{n,k} - s_{n,k} < 1/n, \tag{3.3}$$

$$\max_{1 \le k \le K_n} V(\Delta_{n,k}) < 1/n, \tag{3.4}$$

$$\Delta_{n,k} \cap \Delta_{n,j} = \emptyset \text{ for } k \ne j, \tag{3.5}$$

Moreover, the partitions are nested in the sense that

$$\Delta_{n,k} \cap \Delta_{n+1,j} = \emptyset \text{ or } \Delta_{n+1,j}. \tag{3.6}$$

The non-anticipating stochastic derivative can be represented as the limit [DE10, Theorem 3.1]

$$\mathscr{D}\xi = \lim_{n \to \infty} \phi_n \tag{3.7}$$

with convergence in $\mathcal{I}$ of the stochastic functions of type (2.1) given by

$$\phi_n(t,z) := \sum_{k=1}^{K_n} \mathbb{E}\Big[\xi \frac{\mu(\Delta_{n,k})}{\Lambda(\Delta_{n,k})}\Big| \mathcal{G}_{s_{n,k}}\Big] \mathbf{1}_{\Delta_{n,k}}(t,z) \tag{3.8}$$

where $\Delta_{n,k} = (s_{n,k}, t_{n,k}] \times Z_{n,k}$ refers to the partion of A_n described in (3.1)-(3.6). We have the following result [DE10, Theorem 3.1]:

Theorem 3.1. *All $\xi \in L_2(\Omega, \mathcal{G}, \mathbb{P})$ have representation*

$$\xi = \xi_0 + \int_0^T \int_{\mathcal{Z}} \mathscr{D}_{t,z}\xi \, \mu(dt, dz). \tag{3.9}$$

Moreover $\mathscr{D}\xi_0 = 0$ and $\xi_0 \in L_2(\Omega, \mathcal{G}, \mathbb{P})$ is orthogonal to space generated by $\{I(\phi), \phi \in \mathcal{I}\}$.

[1]Here it is possible to substitute $1/n$ with any sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$.

Indeed, by the orthogonality of the sum in (3.9), one can see that the following duality rule is verified: Let $\xi \in L_2(\Omega, \mathcal{G}, \mathbb{P})$ and $\kappa \in \mathcal{I}$, then

$$\begin{aligned}
&\mathbb{E}\Big[\xi \int_0^T \int_{\mathcal{Z}} \kappa(s,z)\ \mu(ds,dz)\Big] \\
&\quad = \mathbb{E}\Big[\Big(\xi_0 + \int_0^T \int_{\mathcal{Z}} \mathscr{D}_{s,z}\xi\, \mu(ds,dz)\Big) \int_0^T \int_{\mathcal{Z}} \kappa(s,z)\, \mu(ds,dz)\Big] \\
&\quad = \mathbb{E}\Big[\int_0^T \int_{\mathcal{Z}} (\mathscr{D}_{s,z}\xi)\kappa(s,z)\, \Lambda(ds,dz)\Big]. \qquad (3.10)
\end{aligned}$$

4. Optimization problem

Define the state process X_t, $t \in [0,T]$ by $X_0 = a \in \mathbb{R}$ and

$$X_t^{(u)} = X_0 + \int_0^t b_s(u_s, X_s)\, ds + \int_0^t \int_{\mathcal{Z}} \phi_s(z, u_s, X_s)\, \mu(ds,dz).$$

Here $b : \Omega \times [0,T] \times \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ and $\phi : \Omega \times [0,T] \times \mathcal{Z} \times \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ such that $\phi \in \mathcal{I}$. In particular b and ϕ are $\mathbb{G}$-predictable so that X is $\mathbb{G}$-adapted. We assume that X has an unique strong solution. The stochastic process u is the control taking values in an open and convex set $\mathcal{U} \subseteq \mathbb{R}^n$.

In the performance functional (1.1),

$$J(u) = \mathbb{E}\Big[\int_0^T f_t(u_t, X_t)\, dt + g(X_T)\Big], \qquad (4.1)$$

we have $f : \Omega \times [0,T] \times \mathcal{U} \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$. Remark that we have allowed for g and f to depend on additional randomness besides u and X.

We assume f and b are continuously differentiable in $x \in \mathbb{R}$ and $u \in \mathcal{U}$ for all $t \in [0,T]$ and almost all $\omega \in \Omega$. We denote these derivatives $\frac{\partial f_s}{\partial x}$, $\frac{\partial f_s}{\partial u}$, similarly for b and ϕ. Remark that $\frac{\partial f_s}{\partial u} \in \mathbb{R}^n$ since u is n-dimensional. We will denote $\cdot$ as the inner product in $\mathbb{R}^n$ when appropiate. Furthermore g is continuously differentiable with respect to $x \in \mathbb{R}$ a.s., and we denote this derivative by g'.

The random field ϕ is continuously differentiable in $x \in \mathbb{R}$ and $u \in \mathcal{U}$ for almost all $(\omega, t, z) \in \Omega \times [0,T] \times \mathcal{Z}$. We assume that $\frac{\partial \phi}{\partial x} \in \mathcal{I}$ and with $u = (u^1, \ldots, u^n) \in \mathbb{R}^n$, $\frac{\partial \phi}{\partial u^j} \in \mathcal{I}$ for $j = 1, \ldots n$. Finally

$$M_s := \int_0^s \frac{\partial b_r}{\partial x}(u_r, X_r)\, dr + \int_0^s \int_{\mathcal{Z}} \frac{\partial \phi_r}{\partial x}(u_r, X_r)\, \mu(dr,dz), \quad t \in [0,T],$$

has an unique solution which is a semi-martingale.

The *first variation process* $G_s(t)$, $s \in [0,T]$, is the solution to the of the equation

$$G_s(t) := 1 + \int_t^s G_r(t)\, dM_r, \quad s \in [t,T],$$

$$= 1 + \int_t^s G_r(t) \frac{\partial b_r}{\partial x}(u_r, X_r)\, dr + \int_t^s \int_{\mathcal{Z}} G_r(t) \frac{\partial \phi_r}{\partial x}(z, u_r, X_r)\, \mu(dr, dz) \tag{4.2}$$

The solution of (4.2) is given as follows ([Pro05, Theorem II.37])

$$G_s(t) = \exp\Big\{ M_s(t) - \frac{1}{2}[M(t), M(t)]_s \Big\} \prod_{t<s\le T} \big(1 + \Delta M_s(t)\big) \exp\{-\Delta M_s(t)\}$$

where $M(t)$ is the $\mathbb{G}$-semi-martingale defined by $M_s(t) = \int_t^s dM_r$ for $t < s \le T$ and $M_s(t) = 0$ for $s \le t$. Furthermore we define, where $t \in [0,T]$,

$$K_t := K_t^{(u,X)} = g'(X_T)) + \int_t^T \frac{\partial f_s}{\partial x}(u_s, X_s)\, ds, \tag{4.3}$$

$$\mathscr{D}_{t,z} K_t := \mathscr{D}_{t,z} g'(X_T)) + \mathscr{D}_{t,z}\Big(\int_t^T \frac{\partial f_s}{\partial x}(u_s, X_s)\, ds \Big), \tag{4.4}$$

$$F_t(u, X_t) = K_t \frac{\partial b_t}{\partial x}(u_t, X_t) + \int_{\mathcal{Z}} (\mathscr{D}_{t,z} K_t) \frac{\partial \phi_t}{\partial x}(z, u_t, X_t)\, \lambda_t(dz), \tag{4.5}$$

$$p_t := p_t^{(u,X)} = K_t + \int_t^T F_s(u_s, X_s) G_s(t)\, ds, \tag{4.6}$$

$$\kappa_t := \kappa_t^{(u,X)} = \mathscr{D}_{t,z} p_t. \tag{4.7}$$

In order to have the above quantities well-defined the following requirements are needed

Assumption 4.1. The control u with state process $X^{(u)}$ satisfies

$$\mathbb{E}\big[g'(X_T)^2\big] < \infty, \tag{4.8}$$

$$\mathbb{E}\Big[\int_0^T \frac{\partial f_t}{\partial x}(u_t, X_t)^2\, dt\Big] < \infty, \tag{4.9}$$

$$\mathbb{E}\Big[\int_t^T \big(F_s G_s(t)\big)^2\, ds\Big] < \infty, \quad \text{for all } t \in \ [0,T]. \tag{4.10}$$

Remark 4.2. If using the duality relation of Malliavin calculus (4.8)-(4.9)-(4.10) would be stated in terms of Malliavin differentiability, see [MBØZ12, Equation 3.5]. Meaning that both $g'(X_T)$ and $\int_t^T \big(F_s G_s(t)\big)^2\, ds$ need to be in the domain of the Malliavin derivative, a space strictly smaller than $L_2(\Omega, \mathcal{G}, \mathbb{P})$. In addition, (4.9) would be replaced by Malliavin differentiability of $\frac{\partial f_t}{\partial x}(u_t, X_t)$ and integrability of $D_t \frac{\partial f_t}{\partial x}(u_t, X_t)$ so that $\int_0^T D_t \frac{\partial f_t}{\partial x}(u_t, X_t)\, dt$ is well defined (where D is the Malliavin derivative), since the arguments in the forthcoming (5.9) does not apply.

For a given control u with state process $X = X^{(u)}$, we define the Hamiltonian by

$$\begin{aligned} \mathcal{H}_t(v,x) &= \mathcal{H}_t^{(u,X)}(v,x) \\ &:= f_t(v,x) + b_t(v,x) p_t^{(u,X)} + \int_{\mathcal{Z}} \kappa_t^{(u,X)}(z) \phi_t(z,v,x)\, \lambda_t(dz), \end{aligned} \tag{4.11}$$

where $t \in [0,T]$, $v \in \mathcal{U}$ and $x \in \mathbb{R}$.

5. Maximum principle

Let $\mathbb{F} := \{\mathcal{F}_t, t \in [0,T]\}$ be a right continuous filtration such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \in [0,T]$. We state the optimization result for $\mathbb{F}$, naturally we can have $\mathbb{F} = \mathbb{G}$.

Definition 5.1. *We say that u is an admissible control if $u : [0,T] \times \Omega \to \mathcal{U}$ is $\mathbb{F}$-predictable, assumptions 4.1 hold and*

$$\mathbb{E}\Big[\int_0^T f_t(u_t, X_t)^2\, dt + g(X_T)^2 \Big] < \infty. \tag{5.1}$$

We denote the set of admissible controls by $\mathcal{A}^{\mathcal{F}}$.

Assumption 5.2. The following conditions are assumed

i) For all $t, r \in [0,T]$, $t < r \le T$, and bounded $\mathcal{F}_t$-measurable random variables α the control

$$\beta(s) = \alpha(\omega) \mathbf{1}_{(t,r]}(s)$$

belongs to $\mathcal{A}^{\mathcal{F}}$.

ii) For all $u, \beta \in \mathcal{A}^{\mathcal{F}}$ with β bounded there exists $\delta > 0$ such that

$$u + y\beta \in \mathcal{A}^{\mathcal{F}},$$

the family

$$\Big\{ \frac{\partial f_t}{\partial x}\big(u_t + y\beta_t, X_t^{u+y\beta}\big) \frac{\partial}{\partial y} X_t^{u+y\beta_t} + \frac{\partial f_t}{\partial u}\big(u_t + y\beta_t, X^{u+y\beta}\big)\beta_t \Big\}_{y \in (-\delta,\delta)} \tag{5.2}$$

is uniformly $dt \times d\mathbb{P}$-integrable, and the family

$$\Big\{ g'\big(X_T^{u+y\beta}\big) \frac{\partial}{\partial y} X_T^{u+y\beta} \Big\}_{y \in (-\delta,\delta)} \tag{5.3}$$

is uniformly $\mathbb{P}$-integrable.

iii) The process $Y_t^{(u,\beta)} = \frac{\partial}{\partial y} X_t^{u+y\beta}|_{y=0}$ exists as an element of $L_2(\Omega, \mathcal{G}, \mathbb{P})$ for all $t \in [0,T]$ and satisfies

$$
\begin{aligned}
Y_t = Y_t^{(u,\beta)} &= \frac{\partial}{\partial y} X_t^{u+y\beta}\big|_{y=0} \\
&= \int_0^t \big[\frac{\partial b_s}{\partial x}(u_s, X_s) Y_s + \frac{\partial b_s}{\partial u}(u_s, X_s) \cdot \beta_s\big]\, ds \\
&\quad + \int_0^t \int_{\mathcal{Z}} \big[\frac{\partial \phi_s}{\partial x}(z, u_s, X_s) Y_s + \frac{\partial \phi_s}{\partial u}(z, u_s, X_s) \cdot \beta_s\big]\, \mu(ds, dz).
\end{aligned} \tag{5.4}
$$

Theorem 5.3. *Suppose Assumption 5.2 holds. Let $\hat{u}$ be an admissible control. Denote*

$$
\begin{aligned}
\hat{X}_t &= X_t^{(\hat{u})} \\
\hat{\mathcal{H}}_t(v, \hat{X}_t) &= f_t(v, \hat{X}_t) + b_t(\lambda_t, v, \hat{X}_t)\hat{p}_t + \\
&\quad + \int_{\mathcal{Z}} \hat{\kappa}_t(z) \phi_t(z, v, \hat{X}_t)\, \lambda_t(dz), \quad v \in \mathcal{U} \subseteq \mathbb{R}
\end{aligned}
$$

with

$$
\begin{aligned}
\hat{p}_t &= p_t^{(\hat{u}, \hat{X})}, \\
\hat{\kappa}_t &= \kappa_t^{(\hat{u}, \hat{X})}.
\end{aligned}
$$

If $\hat{u}$ is a critical point for $J(u)$, in the sense that

$$
\frac{\partial}{\partial y} J(\hat{u} + y\beta)\big|_{y=0} = 0 \quad \textit{for all bounded } \beta \in \mathcal{A}^{\mathcal{F}},
$$

then

$$
\mathbb{E}\Big[\frac{\partial \mathcal{H}_t}{\partial u}(\hat{u}_t, \hat{X}_t)\Big|\mathcal{F}_t\Big] = 0, \quad dt \times d\mathbb{P}\textit{-a.e.} \tag{5.5}
$$

Conversely, if $\hat{u}$ satisfies (5.5) *then $\hat{u}$ is a critical point.*

For ease of notation we use the short hand notation $b_s = b_s(\hat{u}_s, \hat{X}_s)$, $f_s = f_s(\hat{u}_s, \hat{X}_s)$, and similarly for the other coefficients. We proceed using similar arguments as done in [MBØZ12] with Malliavin derivatives.

Proof. Suppose $\hat{u}$ is a critical point. Then

$$
\begin{aligned}
0 &= \frac{\partial}{\partial y} J(\hat{u} + y\beta)\big|_{y=0} \\
&= \mathbb{E}\big[\int_0^T \frac{\partial f_s}{\partial x} Y_s + \frac{\partial f_s}{\partial u} \cdot \beta_s\, ds + g'(X_T) Y_T\Big].
\end{aligned} \tag{5.6}
$$

By the duality formula (3.10) (and (4.8))

$$
\begin{aligned}
&\mathbb{E}\Big[g'(X_T)Y_T\Big] \\
&= \mathbb{E}\Big[\int_0^T g'(X_T)\big[\frac{\partial b_s}{\partial x}Y_s + \frac{\partial b_s}{\partial u}\cdot\beta_s\big]ds \\
&\qquad + \int_0^T\int_{\mathcal{Z}} \big[\big(\mathscr{D}_{s,z}g'(X_T)\big)\big(\frac{\partial\phi_s}{\partial x}(z)Y_s + \frac{\partial\phi_s}{\partial u}(z)\cdot\beta_s\big)\,\big]\Lambda(ds,dz)\Big].
\end{aligned}
\tag{5.7}
$$

By the Fubini theorem and the duality formula (3.10) (with integrability ensured by (5.2) and the non-anticipating stochastic derivative is well defined by (4.9))

$$
\begin{aligned}
&\mathbb{E}\big[\int_0^T \frac{\partial f_t}{\partial x}Y_t\,dt\big] \\
&= \int_0^T \mathbb{E}\Big[\frac{\partial f_t}{\partial x}\big[\int_0^t \frac{\partial b_s}{\partial x}Y_s + \frac{\partial b_t}{\partial u}\cdot\beta_s\,ds\big] \\
&\qquad + \frac{\partial f_t}{\partial x}\big[\int_0^t\int_{\mathcal{Z}} \big(\frac{\partial\phi_s}{\partial x}Y_s + \frac{\partial\phi_s}{\partial u}\cdot\beta_s\big)\,\mu(ds,dz)\big]\Big]\,dt \\
&= \mathbb{E}\Big[\int_0^T \Big\{\frac{\partial f_t}{\partial x}\big[\int_0^t \frac{\partial b_s}{\partial x}Y_s + \frac{\partial b_s}{\partial u}\cdot\beta_s\,ds\big] \\
&\qquad + \big[\int_0^t\int_{\mathcal{Z}} \big(\mathscr{D}_{s,z}\frac{\partial f_t}{\partial x}\big)\big(\frac{\partial\phi_s}{\partial x}Y_s + \frac{\partial\phi_s}{\partial u}\cdot\beta_s\big)\,\Lambda(ds,dz)\big]\Big\}\,dt\Big] \\
&= \mathbb{E}\Big[\int_0^T \big[\int_t^T \frac{\partial f_s}{\partial x}\,ds\big]\big(\frac{\partial b_t}{\partial x}Y_t + \frac{\partial b_t}{\partial u}\cdot\beta_t\big)\,dt \\
&\qquad + \int_0^T\int_{\mathcal{Z}} \big[\int_t^T \mathscr{D}_{t,z}\frac{\partial f_s}{\partial x}\,ds\big]\big(\frac{\partial\phi_t}{\partial x}Y_t + \frac{\partial\phi_t}{\partial u}\cdot\beta_t\big)\,\Lambda(dt,dz)\Big].
\end{aligned}
\tag{5.8}
$$

By the continuity of $\mathscr{D}$ [DE10, Remark 3.4] and with sufficent integrability from (4.9) we have

$$
\int_t^T \mathscr{D}_{t,z}\frac{\partial f_s}{\partial x}\,ds = \mathscr{D}_{t,z}\int_t^T \frac{\partial f_s}{\partial x}\,ds, \quad d\Lambda\times d\mathbb{P}\text{ a.e.}
\tag{5.9}
$$

We recall (4.3), (4.4), and by (5.6)-(5.7)-(5.8) conclude that

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T K_s\Big(\frac{\partial b_s}{\partial x}Y_s+\frac{\partial b_s}{\partial u}\cdot\beta_s\Big)+\frac{\partial f_s}{\partial u}\cdot\beta_s\,ds\\ &+\int_0^T\int_{\mathcal{Z}}(\mathscr{D}_{s,z}K_s)\Big(\frac{\partial \phi_s}{\partial x}Y_s+\frac{\partial \phi_s}{\partial u}\cdot\beta_s\Big)\,\Lambda(ds,dz)\Big]=0.\end{aligned}\tag{5.10}$$

Set $\beta_s=(\alpha^{(1)},\dots,\alpha^{(n)})\mathbf{1}_{(t,t+h]}(s)$, where $0\le t<t+h\le T$ and $\alpha^{(j)}$, $1\le j\le n$ are bounded and $\mathcal{G}_t$-measurable random variables. Then $Y_s=Y_s^{(u,\beta)}=0$ for $s<t$ so that (5.10) can be rewritten as

$$A_1+A_2=0\tag{5.11}$$

where

$$\begin{aligned}A_1&=\mathbb{E}\Big[\int_t^T K_s\frac{\partial b_s}{\partial x}Y_s\,ds+\int_t^T\int_{\mathcal{Z}}(\mathscr{D}_{s,z}K_s)\frac{\partial \phi_s}{\partial x}Y_s\,\Lambda(ds,dz)\Big],\\ A_2&=\mathbb{E}\Big[\alpha\cdot\Big(\int_t^{t+h}\big[K_s\frac{\partial b_s}{\partial u}+\frac{\partial f_s}{\partial u}\big]\,ds+\int_t^{t+h}\int_{\mathcal{Z}}(\mathscr{D}_{s,z}K_s)\frac{\partial \phi_s}{\partial u}\,\Lambda(ds,dz)\Big)\Big].\end{aligned}$$

From (4.5)

$$\begin{aligned}A_1&=\mathbb{E}\Big[\int_t^T F_sY_s\,ds\Big]\\ &=\int_t^{t+h}\mathbb{E}\Big[F_sY_s\Big]ds+\int_{t+h}^T\mathbb{E}\Big[F_sY_s\Big]\,ds.\end{aligned}$$

Since Y admits a càdlàg representative and $Y_t=0$ we have

$$\frac{\partial}{\partial h}\int_t^{t+h}\mathbb{E}\Big[F_sY_s\Big]ds\Big|_{h=0}=0.$$

Recall (5.4) and (4.2). We have

$$Y_s=Y_{t+h}G_s(t+h)\quad\text{for } s\ge t+h.$$

Since $Y_t = 0$ (interchange of integration and expectation justified by (5.2), (5.3))

$$\begin{aligned}
\frac{\partial}{\partial h} A_1\big|_{h=0} &= \frac{\partial}{\partial h} \int_{t+h}^{T} \mathbb{E}\Big[F_s Y_s\Big]\, ds \;\big|_{h=0} \\
&= \int_t^T \frac{\partial}{\partial h}\Big\{\mathbb{E}\Big[F_s Y_{t+h} G_s(t+h)\Big]\Big\}\, ds\Big|_{h=0} - F_t Y_t \\
&= \int_t^T \mathbb{E}\Big[F_s \Big\{Y_{t+h} \frac{\partial}{\partial h} G_s(t+h) + G_s(t+h) \frac{\partial}{\partial h} Y_{t+h}\Big\}\Big]\, ds\Big|_{h=0} \\
&= \int_t^T \frac{\partial}{\partial h} \mathbb{E}\Big[F_s G_s(t) Y_{t+h}\Big]\Big|_{h=0}\, ds.
\end{aligned}$$

By (5.4) we have

$$\begin{aligned}
Y_{t+h} = \alpha \cdot \Big(&\int_t^{t+h} \frac{\partial b_s}{\partial u}\, ds + \int_t^{t+h} \int_{\mathcal{Z}} \frac{\partial \phi_s}{\partial u}\, \mu(ds, dz)\Big) \\
&+ \int_t^{t+h} Y_s \frac{\partial b_s}{\partial x}\, ds + \int_t^{t+h} \int_{\mathcal{Z}} Y_s \frac{\partial \phi_s}{\partial x}\, \mu(ds, dz).
\end{aligned}$$

Denote $\frac{\partial}{\partial h} A_1|_{h=0} = B_1 + B_2$ with

$$B_1 = \int_t^T \frac{\partial}{\partial h} \mathbb{E}\Big[F_s G_s(t) \Big\{\alpha \cdot \Big(\int_t^{t+h} \frac{\partial b_r}{\partial u}\, dr + \int_t^{t+h} \int_{\mathcal{Z}} \frac{\partial \phi_r}{\partial u}\, \mu(dr, dz)\Big)\Big\}\Big]\Big|_{h=0}\, ds,$$

$$B_2 = \mathbb{E}\Big[\int_t^T \frac{\partial}{\partial h} \mathbb{E}\Big[F_s G_s(t) \Big\{ \int_t^{t+h} Y_r \frac{\partial b_r}{\partial x}\, dr + \int_t^{t+h} \int_{\mathcal{Z}} Y_r \frac{\partial \phi_r}{\partial x}\, \mu(dr, dz)\Big\}\Big]\Big|_{h=0}\, ds.$$

By the duality formula (3.10) (well defined by (4.10))

$$\begin{aligned}
B_1 &= \int_t^T \frac{\partial}{\partial h} \mathbb{E}\Big[F_s G_s(t) \Big\{\alpha \cdot \Big(\int_t^{t+h} \frac{\partial b_r}{\partial u}\, dr + \int_t^{t+h} \int_{\mathcal{Z}} \frac{\partial \phi_r}{\partial u}\, \mu(dr, dz)\Big)\Big\}\Big]\Big|_{h=0}\, ds \\
&= \int_t^T \frac{\partial}{\partial h} \mathbb{E}\Big[\Big\{\alpha \cdot \Big(\int_t^{t+h} F_s G_s(t) \frac{\partial b_r}{\partial u}\, dr \\
&\quad + \int_t^{t+h} \int_{\mathcal{Z}} \mathscr{D}_{r,z}\big(F_s G_s(t)\big) \frac{\partial \phi_r}{\partial u}\, \lambda_r(dz)\, dr\Big)\Big\}\Big]\Big|_{h=0}\, ds
\end{aligned}$$

(5.12)

$$= \int_t^T \mathbb{E}\Big[\Big\{\alpha \cdot \Big(F_s G_s(t)\frac{\partial b_t}{\partial u} + \int_{\mathcal{Z}} \mathscr{D}_{t,z}\big(F_s G_s(t)\big)\frac{\partial \phi_t}{\partial u}\,\lambda_t(dz)\Big)\Big\}\Big]\,ds.$$

By the duality formula (3.10) (well defined by (4.10)) and since $Y_t = 0$ we have

$$\begin{aligned}
B_2 &= \int_t^T \frac{\partial}{\partial h}\mathbb{E}\Big[F_s G_s(t)\Big\{\int_t^{t+h} Y_r \frac{\partial b_r}{\partial x}\,dr + \int_t^{t+h}\int_{\mathcal{Z}} Y_r \frac{\partial \phi_r}{\partial x}\,\mu(dr,dz)\Big\}\Big]\Big|_{h=0}\,ds \\
&= \int_t^T \mathbb{E}\Big[\frac{\partial}{\partial h}\Big\{\int_t^{t+h} F_s G_s(t) Y_r \frac{\partial b_r}{\partial x}\,dr \\
&\quad + \int_t^{t+h}\int_{\mathcal{Z}} \mathscr{D}_{r,z}\big(F_s G_s(t)\big) Y_r \frac{\partial \phi_r}{\partial x}\,\lambda_r(dz)\,dr\Big\}\Big]\Big|_{h=0}\,ds \\
&= \int_t^T \mathbb{E}\Big[\Big\{F_s G_s(t) Y_t \frac{\partial b_t}{\partial x} + \int_{\mathcal{Z}} \mathscr{D}_{r,z}\big(F_s G_s(t)\big) Y_t \frac{\partial \phi_t}{\partial x}\,\lambda_t(dz)\Big\}\Big]\,ds
\end{aligned}$$

(5.13) $= 0.$

We see immediately that (interchange of derivation and expectation justified by (5.2) (5.3))

$$\frac{\partial}{\partial h}A_2\Big|_{h=0} = \mathbb{E}\Big[\alpha \cdot \Big(K_t \frac{\partial b_t}{\partial u} + \frac{\partial f_t}{\partial u} + \int_{\mathcal{Z}} (\mathscr{D}_{t,z}K_t)\frac{\partial \phi_t}{\partial u}\,\lambda_t(dz)\Big)\Big]. \tag{5.14}$$

Recall that $\frac{\partial}{\partial h}A_1 = B_1 + B_2$ and the definition of p in (4.6). By (5.12)-(5.13)-(5.14) we have

$$\begin{aligned}
\frac{\partial}{\partial h}\{A_1 + A_2\}_{h=0} &= \mathbb{E}\Big[\alpha \cdot \Big\{\frac{\partial f_t}{\partial u} + p_t \frac{\partial b_t}{\partial u} + \int_{\mathcal{Z}} \big(\mathscr{D}_{t,z}p_t\big)\frac{\partial \phi_t}{\partial u}(z)\,\lambda_t(dz)\Big\}\Big] \\
&= \mathbb{E}\Big[\alpha \cdot \frac{\partial \mathcal{H}_t}{\partial u}(\hat{X}_t, \hat{u}_t)\Big].
\end{aligned}$$

As a function of h, $A_1(h) + A_2(h) = 0$ for all $0 \le h \le T - t$ by (5.11). Hence $\frac{\partial}{\partial h}\{A_1(h) + A_2(h)\} = 0$. Since this holds for all bounded $\mathcal{F}_t$-measurable α we have

$$\mathbb{E}\Big[\frac{\partial \mathcal{H}_t}{\partial u}(\hat{X}_t, \hat{u}_t)\Big|\mathcal{F}_t\Big] = 0.$$

This complete the proof for the sufficient condition.

Conversely, suppose (5.5). By reversing the above argument we get that (5.11) holds for all $\beta \in \mathcal{A}^{\mathcal{F}}$ of the form

$$\beta(s,\omega) = \alpha(\omega)\mathbf{1}_{(t,t+h]}(s)$$

where random variable $\alpha \in \mathbb{R}^n$ is $\mathcal{F}_t$-measurable and bounded and $0 \le t < t+h \le T$. Hence (5.11) holds for all linear combinations of such β. Since any $\beta \in \mathcal{A}^{\mathcal{F}}$ can be approximated by such linear combinations it follows that (5.11) holds for all bounded $\beta \in \mathcal{A}^{\mathcal{F}}$. □

6. Lévy Processes

Here we compare the non-anticipating stochastic derivative with the Malliavin operator in the case of Lévy processes. We refer to e.g. [DR07, DØP09, Nua95, SUV07] for a comprehensive treatment on Malliavin calculus on Lévy processes, or more precisely the mixture of Gaussian and Poisson random measures. In short, the stochastic non-anticipating derivative coincide with the projection of the Malliavin derivative D

$$\mathscr{D}_{s,z}\xi = \mathbb{E}\big[D_{s,z}\xi \big| \mathcal{G}_s\big] \tag{6.1}$$

whenever the right hand side is well defined. But the domain of the Malliavin derivative D is $\mathbb{D}_{1,2}$, a space strictly smaller than $L_2(\Omega, \mathcal{G}, \mathbb{P})$.

Let N be a Poisson random field on $[0,T] \times \mathbb{R}_0$ with expectation $\nu(dz)dt$ and denote $\tilde{N} = N - \nu$ as the centered Poisson random field. Furthermore B_t, $t \in [0,T]$ is a Brownian Motion. With $\mathbb{G}$ as the (completed) filtration generated by B and N, let the martingale random field μ be given by :

$$\mu(dt, dz) = \mathbf{1}_{\{0\}}(z) dB_t + \mathbf{1}_{\mathbb{R}_0}(z)\, \tilde{N}(dt, dz),$$

with $\lambda_t(dz) = \mathbf{1}_{\{0\}}(z) + \mathbf{1}_{\mathbb{R}_0}(z)\nu(dz)$. Then the process η defined by

$$\eta_t = B_t + \int_0^t \int_{\mathbb{R}_0} z\, \tilde{N}(ds, dz), \quad t \in [0,T],$$

is a Lévy process. If we assume that $(\Omega, \mathcal{G}, \mathbb{P}) = (\Omega, \mathcal{G}_T, \mathbb{P})$, i.e. that our probability space can be generated by B and N, then there exists a subspace $\mathbb{D}_{1,2} = \mathbb{D}^B_{1,2} \cap \mathbb{D}^N_{1,2} \subsetneq L_2(\Omega, \mathcal{G}, \mathbb{P})$ such that for any $\xi \in \mathbb{D}_{1,2}$ [BDNL+03, Theorem 3.11]

$$\xi = \mathbb{E}\big[\xi\big] + \int_0^T \mathbb{E}[D_s\xi | \mathcal{G}_s]\, dB_s + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{s,z}\xi | \mathcal{G}_s]\, \tilde{N}(ds, dz) \tag{6.2}$$

where D_s and $D_{s,z}$ are the Mallavin derivatives for the Brownian motion and Poisson random field with domains $\mathbb{D}^B_{1,2}$ and $\mathbb{D}^N_{1,2}$ respectively. This is also known as the Clark-Ocone formula. Furthermore $D_s\xi \in L_2(\Omega \times [0,T])$ and $D_{s,z} \in L_2(\Omega \times [0,T] \times \mathbb{R}_0)$.

Remark that the Clark-Ocone formula (6.2) can be extended to $L_2(\Omega, \mathcal{G}, \mathbb{P})$ in the setting of white noise and Hida-Malliavin derivatives, see [AØPU00, DØP04]. However this requires further assumptions on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$.

7. Application to default risk

Here we show an application of the maximum principle to portfolio optimization. We chose a setting outside Lévy processes that has independent interest, assets with credit risk modeled by doubly stochastic Poisson processes. Credit risk with doubly stochastic Poisson processes has been widely studied in the literature, see e.g. [JY01, Lan98, Duf05].

Let $\lambda_s = (\lambda_s^{(1)}, \dots \lambda_s^{(n)})$, $s \in [0,T]$, be a positive, stochastic process in $\mathbb{R}^n$. Let $\Lambda_t^{(j)} = \int_0^t \lambda_s^{(j)}\, ds$, and denote the filtration generated by λ as $\mathbb{F}^\Lambda = \{\mathcal{F}_t^\Lambda,\ t \in [0,T]\}$. No assumptions of independence are required between $\Lambda^{(j)}$ and $\Lambda^{(k)}$ for any $j \neq k$.

The n-dimensional pure jump process $H_s = (H_s^{(1)}, \dots, H_s^{(n)})$ is a doubly stochastic Poisson process if, when conditioned on the λ's, it is Poisson distributed. We assume that

$$\mathbb{P}\big(H_t^{(j)} = k \,\big|\, \mathcal{F}_T^\Lambda\big) = \mathbb{P}\big(H_t^{(j)} = k \,\big|\, \Lambda_t^{(j)}\big) = \frac{(\Lambda_t^{(j)})^k}{k!} e^{-\Lambda_t^{(j)}}$$

for all $1 \leq j \leq n$ and $k \in \mathbb{N}$. Let $\tilde{H}_t := H_t - \Lambda_t$, $t \in [0,T]$ and $\mathbb{F} := \{\mathcal{F}_t, t \in [0,T]\}$ the filtration generated by $\tilde{H}$. Let $\mathbb{G}$ be such that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_T^\lambda$ and $\mathcal{Z} = \{1, \dots n\}$, where $\mathcal{Z}$ is equipped with the discrete topology. Then μ defined by $\mu(dt, z) = d\tilde{H}_t^{(z)}$ is a martingale random field with respect to both $\mathbb{F}$ and $\mathbb{G}$ on $[0,T] \times \mathcal{Z}$.

The filtration $\mathbb{G}$ contains anticipating information in knowing future values of λ. It is however a natural candidate for computing the non-anticipating stochastic derivative for doubly stochastic Poisson processes, which has been studied in [DS13]. Computational rules of Malliavin type can also be found in [Yab07].

Let $\tau^{(z)}$ be the first jump of $H^{(z)}$, $z = 1, \dots n$. We model each asset $S^{(z)}$ with a return $\rho^{(z)} + \lambda^{(z)}$ up to the time of default τ. In the case of default the asset $S^{(z)}$ becomes worthless, i.e. $S_{\tau^{(z)}}^{(z)} = 0$ (whenever $\tau^{(z)} < T$). The goal is to maximize expected utility of the wealth at the terminal time T by investing in the n assets In mathematical terms: Let

$$\begin{aligned}
S_t^{(1)} &= S_{t-}^{(1)} \mathbf{1}_{\{\tau^{(1)} > t\}}(t) \big(\rho_t^{(1)}\, dt - d\tilde{H}_t^{(1)}\big), \\
&\vdots \\
S_t^{(n)} &= S_{t-}^{(n)} \mathbf{1}_{\{\tau^{(n)} > t\}}(t) \big(\rho_t^{(n)}\, dt - d\tilde{H}_t^{(n)}\big).
\end{aligned}$$

Let X denote the total wealth of the investor and the control u denote the amount invested in the n assets:

$$X_t = \int_0^t \sum_{z=1}^n \mathbf{1}_{\{\tau^{(z)} > r\}}(r) u_r^{(z)} \rho_r^{(z)}\, dr - \int_0^t \sum_{z=1}^n \mathbf{1}_{\{\tau^{(z)} > r\}}(r) u_r^{(z)}\, d\tilde{H}_t^{(z)}.$$

Remark that every asset $S^{(z)}$ and the wealth process X are $\mathbb{F}$-adapted. Hence $\mathbb{F}$ is a natural model for the investor's information. With

$$J(u) = \mathbb{E}\Big[U(X_T)\Big] \tag{7.1}$$

where $U : \mathbb{R} \to \mathbb{R}$ is an utility function (increasing and concave), we look for

$$\sup_{u \in \mathcal{A}^{\mathcal{F}}} J(u). \tag{7.2}$$

We have

$$\begin{aligned}
K_t &= U'(X_T) \\
F_t &= U'(X_T) \sum_{z=1}^{n} \mathbf{1}_{\{\tau^{(z)}>t\}}(t) u_t^{(z)} \rho_t^{(z)} + \sum_{z=1}^{n} \big(\mathscr{D}_{t,z} U'(X_t)\big) \mathbf{1}_{\{\tau^{(z)}>t\}}(t) u_t^{(z)} \lambda_t^{(z)} \\
p_t &= U'(X_T) \\
\kappa_t &= \sum_{z=1}^{n} \mathscr{D}_{t,z} U'(X_t) \\
G_s(t) &= 0,
\end{aligned}$$

so the Hamiltonian (4.11) is given by

$$\begin{aligned}
\mathcal{H}_t(u,x) = U'(X_T) &\sum_{z=1}^{n} \mathbf{1}_{\{\tau^{(z)}>t\}}(t) u_t^{(z)} \rho_t^{(z)} \\
&+ \sum_{z=1}^{n} \big(\mathscr{D}_{t,z} U'(X_t)\big) \mathbf{1}_{\{\tau^{(z)}>t\}}(t) u_t^{(z)} \lambda_t^{(z)}.
\end{aligned}$$

Hence

$$\frac{\partial \mathcal{H}_t}{\partial u}(v,x) = U'(X_T) \sum_{z=1}^{n} \mathbf{1}_{\{\tau^{(z)}>t\}}(t) \rho_t^{(z)} + \sum_{z=1}^{n} \big(\mathscr{D}_{t,z} U'(X_T)\big) \mathbf{1}_{\{\tau^{(z)}>t\}}(t) \lambda_t^{(z)}.$$

Theorem 5.3 finds critical points for (7.1). To ensure that a critical point $\hat{u}$ is a solution to (7.2) we need to know that 1) the critical point is a local maximum and 2) there are no other critical points $\bar{u}$ where $J(\bar{u}) > J(\hat{u})$. We investigate the exact properties of the critical points in Proposition 7.1 and sufficent conditions for a solution to (7.2) is given in Corollary 7.2.

Proposition 7.1. *Assume that*

i) *U is twice continuously differentiable and concave,*

ii) *for any* $u \in \mathcal{A}^{\mathcal{F}}$ *and bounded* $\beta \in \mathcal{A}^{\mathcal{F}}$ *there exists* $\epsilon > 0$ *such that the family*

$$\Big\{U''\big(X_T^{u+y\beta}\big)\Big(\int_0^t \sum_{z=1}^n \mathbf{1}_{\{\tau^{(z)}>r\}}(r)\beta^{(z)}\rho_r^{(z)}\,dr - \sum_{z=1}^n \int_0^t \mathbf{1}_{\{\tau^{(z)}>r\}}(r)\beta^{(z)}\,d\tilde{H}_t^{(z)}\Big)^2\Big\}_{y\in(-\varepsilon,\varepsilon)} \tag{7.3}$$

is uniformly $\mathbb{P}$*-integrable,*

iii) *Assumption 5.2 holds.*

Let $\epsilon = min(\delta, \varepsilon)$*, where* δ *is as in* (5.3)*. Then the mapping* $y \to J(u + y\beta)$*,* $y \in (-\epsilon, \epsilon)$*, is concave. Furthermore, there is at most one bounded* $u \in \mathcal{A}^{\mathcal{F}}$ *such that* u *is a critical point (in the sense of Theorem 5.3).*

Proof. First we prove the concavity of the mapping $y \to J(u + y\beta)$. We interchange the derivation and expectation and get

$$\begin{aligned}\frac{\partial^2}{\partial y^2}J(u+y\beta) &= \mathbb{E}\Big[\frac{\partial^2}{\partial y^2}U(X_T^{u+y\beta})\Big]\\ &= \mathbb{E}\Big[U''\big(X^{u+y\beta_T}\big)\Big(\int_0^t \sum_{z=1}^n \mathbf{1}_{\{\tau^{(z)}>r\}}(r)\beta^{(z)}\rho_r^{(z)}\,dr\\ &\qquad - \sum_{z=1}^n \int_0^t \mathbf{1}_{\{\tau^{(z)}>r\}}(r)\beta^{(z)}\,d\tilde{H}_t^{(z)}\Big)^2\Big] < 0,\end{aligned}$$

where the last inequality follows by the concavity of U.

Next we want to show that there is at most one bounded $u \in \mathcal{A}^{\mathcal{F}}$ such that u is a critical point.

It is sufficient to show that when both $u, \beta \in \mathcal{A}^{\mathcal{F}}$ are bounded we have $\epsilon > 1$, i.e. that $y \to J(u + y\beta)$ is a concave mapping for $y \in [1-, 1]$. We can then compare any two bounded controls $\hat{u}, \bar{u} \in \mathcal{A}^{\mathcal{F}}$ by setting $\beta = \bar{u} - \hat{u}$ and conclude that at most one can be a critical point by the concavity of $y \to J(\hat{u} + y\beta)$. The claim $\epsilon > 1$ follows from the uniform integrability conditions (5.3) and (7.3) since $\frac{\partial}{\partial y}J(u+y\beta)|_{y=a} = \frac{\partial}{\partial y}J(u+a\beta+y\beta)|_{y=0}$. $\square$

Corollary 7.2. *If* $\mathcal{U}$ *is bounded and a critical point* $\hat{u}$ *exists, then* $\hat{u}$ *is optimal, i.e.*

$$J(\hat{u}) = \sup_{u\in\mathcal{A}^{\mathcal{F}}} J(u),$$

and optimal portfolio $\hat{u}$ is characterized by

$$\mathbb{E}\big[\frac{\partial \mathcal{H}_t}{\partial u}(\hat{u}, X_t^{(\hat{u})})\big|\mathcal{F}_t\big] = \sum_{z=1}^{n} \mathbf{1}_{\{\tau^{(z)}>t\}}(t)\rho_t^{(z)}\mathbb{E}\big[U'(X_T)\,\big|\mathcal{F}_t\big] + \sum_{z=1}^{n}\big(\mathscr{D}_{t,z}U'(X_T)\big)\mathbf{1}_{\{\tau^{(z)}>t\}}(t)\lambda_t^{(z)} = 0,$$

for all $t \in [0,T]$ a.s.

Proof. This is a restatement of Proposition 7.1. □

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References

[AØ12] Ta Thi Kieu An and Bernt Øksendal. A maximum principle for stochastic differential games with g-expectations and partial information. *Stochastics An International Journal of Probability and Stochastic Processes*, 84(2-3):137–155, 2012.

[AØPU00] Knut Aase, Bernt Øksendal, Nicolas Privault, and Jan Ubøe. White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. *Finance and Stochastics*, 4:465–496, 2000.

[BDNL+03] Fred Espen Benth, Giulia Di Nunno, Arne Løkka, Bernt Øksendal, and Frank Proske. Explicit representation of the minimal variance portfolio in markets driven by Lévy processes. *Mathematical Finance*, 13(1):55–72, 2003.

[Ben92] Alain Bensoussan. *Stochastic control of partially observable systems*. Cambridge University Press Cambridge, 1992.

[BØ07] Fouzia Baghery and Bernt Øksendal. A maximum principle for stochastic control with partial information. *Stochastic Analysis and Applications*, 25(3):705–717, 2007.

[Çın11] Erhan Çınlar. Measure and integration. In *Probability and Stochastics*, volume 261 of *Graduate Texts in Mathematics*, pages 1–47. Springer New York, 2011.

[CW75] Renzo Cairoli and John Walsh. Stochastic integrals in the plane. *Acta Mathematica*, 134:111–183, 1975.

[DE10] Giulia Di Nunno and Inga Baadshaug Eide. Minimal-variance hedging in large financial markets: random fields approach. *Stochastic Analysis and Applications*, 28, 2010.

[Di 02] Giulia Di Nunno. Stochastic integral representation, stochastic derivatives and minimal variance hedging. *Stochastics and Stochastics Reports*, 73:181–198, 2002.

[Di 03] Giulia Di Nunno. Random Fields Evolution: non-anticipating integration and differentiation. *Theory of Probability and Mathematical Statistics, AMS*, 66:91–104, 2003.

[Di 07] Giulia Di Nunno. Random Fields: non-anticipating derivative and differentiation formulas. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 10:465–481, 2007.

[DNØ09] Giulia Di Nunno and Bernt Øksendal. Optimal portfolio, partial information and malliavin calculus. *Stochastics: An International Journal of Probability and Stochastics Processes*, 81(3-4):303–322, 2009.

[DØP04] Giulia Di Nunno, Bernt Øksendal, and Frank Proske. White noise analysis for Lévy processes. *Journal of Functional Analysis*, 206(1):109 – 148, 2004.

[DØP09] Giulia Di Nunno, Bernt Øksendal, and Frank Proske. *Malliavin Calculus for Lévy Processes with Applications to Finance*. Springer, 2009.

[DR07] Giulia Di Nunno and Yuri Rozanov. Stochastic integrals and adjoint derivatives. In Fred Espen Benth, Giulia Di Nunno, Tom Lindstrøm, Bernt Øksendal, and Tusheng Zhang, editors, *Stochastic Analysis and Applications*, volume 2 of *Abel Symposia*, pages 265–307. Springer Berlin Heidelberg, 2007.

[DS13] Giulia Di Nunno and Steffen Sjursen. On chaos representation and orthogonal polynomials for the doubly stochastic Poisson process. In Robert Dalang, Marco Dozzi, and Francesco Russo, editors, *Seminar on Stochastic Analysis, Random Fields and Applications VII*, volume 67 of *Progress in Probability*. Birkhäuser Basel, 2013. To appear.

[Duf05] Darrel Duffie. Credit risk modeling with affine processes. *Journal of Banking & Finance*, 25, 2005.

[JY01] Robert A. Jarrow and Fan Yu. Counterparty risk and the pricing of defaultable securities. *The Journal of Finance*, 56(5):1765–1799, 2001.

[KX91] Ioannis Karatzas and Xlng-Xlong Xue. A note on utility maximization under partial observations1. *Mathematical Finance*, 1(2):57–70, 1991.

[Lan98] David Lando. On Cox processes and credit risky securities. *Review of Derivatives Research*, 2(2-3):99–120, 1998.

[MBØZ12] Thilo Meyer-Brandis, Bernt Øksendal, and Xun Yu Zhou. A mean-field stochastic maximum principle via Malliavin calculus. *Stochastics An International Journal of Probability and Stochastic Processes*, 84(5-6):643–666, 2012.

[Nua95] David Nualart. *The Malliavin Calculus and Related Topics*. Springer, 1995.

[Pro05] Philip Protter. *Stochastic Integration and Differential Equations*. Springer, 2005. Version 2.1.

[SUV07] Josep Lluís Solé, Frederic Utzet, and Josep Vives. Canonical Lévy process and Malliavin calculus. *Stochastic Processes and their Applications*, 117(2):165 – 187, 2007.

[Tan98] Shanjian Tang. The maximum principle for partially observed optimal control of stochastic differential equations. *SIAM Journal on Control and Optimization*, 36(5):1596–1617, 1998.

[Yab07] Aleh L. Yablonski. The Malliavin calculus for processes with conditionally independent increments. In Fred Espen Benth, Giulia Nunno, Tom Lindstrøm, Bernt Øksendal, and Tusheng Zhang, editors, *Stochastic Analysis and Applications*, volume 2 of *Abel Symposia*, pages 641–678. Springer Berlin Heidelberg, 2007.

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